1. Exercises from Sections 2.2-2.3

Problem 1. *(Folland 2.1.8) Suppose $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$. If all partial derivatives $\partial_j f$ exist and are bounded on S, then f is continuous on S.*

Remark: Notice the converse is not true - i.e. $f(x,y) = x^2 + y^2$ on $\mathbb{R}^2$ has unbounded partials

Proof.

- Fix $\epsilon > 0$, pick any $x \in S$, and consider $|f(x_i + h_i) - f(x)|$ for a vector h.
- We proceed by adding zero in a clever way and using the triangle inequality:

$$\begin{aligned} |f(x_1+h_1,\dots,x_n-h_n) - f(x_1,\dots,x_n)| &= |f(x_1+h_1,\dots,x_n+h_n) - f(x_1+h_1,\dots,x_{n-1}+h_{n-1},x_n) \\ &+ \quad f(x_1+h_1,\dots,x_{n-1}+h_{n-1},x_n) - f(x_1,\dots,x_n)| \\ &= \left|\sum_{j=1}^{n} f(x_1+h_1,\dots,x_j+h_j,x_{j+1},\dots,x_n) - f(x_1+h_1,\dots,x_j,x_{j+1},\dots,x \right. \\ &\leq \sum_{j=1}^{n} |f(x_1+h_1,\dots,x_j+h_j,x_{j+1},\dots,x_n) - f(x_1+h_1,\dots,x_j,x_{j+1},\dots,x_n \end{aligned}$$

- For each j, fix all the coordinates where $k \neq j$, then by the mean value theorem for 1 variable functions there exists $c_j \in (x_j, x_j+h_j)$ such that $\frac{\partial f}{\partial x_j}(c_j) = (f(x_j+h_j) - f(x_j))/h_j$

$$\sum_{j=1}^{n} |f(x_1+h_1,\dots,x_j+h_j,x_{j+1},\dots,x_n) - f(x_1+h_1,\dots,x_j,x_{j+1},\dots,x_n)| = \sum_{j=1}^{n} \left|\frac{\partial f}{\partial x_j}(c_j)h_j\right|$$

- Applying boundedness of the partial derivatives, for each j there exists M_j such that $|\partial_{x_j} f(c_j)| < M_j$, then
$$\sum_{j=1}^{n} \left|\frac{\partial f}{\partial x_j}(c_j)h_j\right| \leq \sum_{j=1}^{n} M_j h_j \leq n \max\{M_j\} \sum_j h_j$$
- $\sum_j h_j \to 0$ as $h \to 0$, therefore just pick h small enough that $\sum_j h_j < \epsilon/n \max(N_j)$.

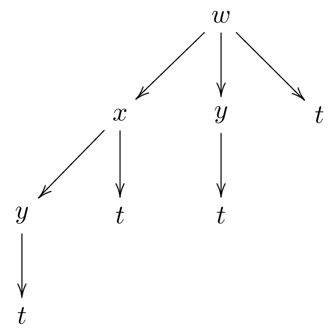
$\square$

Problem 2. *Let $f(x,y) = e^{4x-y^2}$. Compute ∇f and find the directional derivative of f in the direction $u = (\frac{3}{5}, \frac{4}{5})$ at the point $(1,-2)$*

Remember that:
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(4e^{4x-y^2}, -2ye^{4x-y^2}\right)$$
and is interpreted as the direction in which f is most rapidly increasing. We can find the rate of change in a particular direction by taking the inner product of ∇f at the point $(1,-2)$ with the direction u:
$$\partial_u f = \nabla f \cdot u = (4e^0, -2(-2)e^0) \cdot (\frac{3}{5}, \frac{4}{5}) = \frac{12}{5} + \frac{16}{5} = \frac{28}{5}$$

Problem 3. *(Practice with the chain rule) Suppose that $w = f(x,y,t)$, $x = g(y,t)$, $y = h(t)$. Find dw/dt.*

The easiest way to keep your book-keeping straight is to draw the "derivative tree" and just follow the branches.



Now we multiply along each "branch" and add the separate branches:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial t}$$

PROBLEM 4. *Find the tangent plane to the surface S in $\mathbb{R}^3$ described by the following equations at the given point in $a \in \mathbb{R}^3$.*

(1) $z = x^2 - y^3, a = (2, -1, 5)$
(2) $x^2 + 2y^2 + 3z^2 = 6, a = (1, 1, -1)$
(3) $xyz^2 - \log(z-1) = 8, a = (-2, -1, 2)$

Note for Peter: Try and draw an embedded surface in $\mathbb{R}^3$ to give students a "feel" for what tangent spaces really are, then carry out the calculations. Make a joke about how calculus and linear algebra are about to make babies.

The strategy for all of these problems is the same. Theorem 2.37 tells us that if $F : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and we define an embedded surface in $\mathbb{R}^3$ by the level set $F(x, y, z) = c$, then ∇F is perpendicular to S at each point in S. We use this fact to define what the normal vectors of the tangent planes are.

Part 1: We can rearrange to get $g_1(x, y, z) = x^2 - y^3 - z$, which is clearly a differentiable function of x, y, z. Now $\nabla g_1 = 2x\partial_x - 3y^2\partial_y - \partial_z$, so at the point $(2, -1, 5)$ we have that $(\nabla g_1)_{(2,-1,5)} = (4, -3, -1)$. The tangent plane $T_{(2,-1,5)}S_1$ is defined by the equation $4x - 3y - z = d$, and we can solve for d since we know a point on the plane. Plugging in $(2, -1, 5)$ gives $d = 4(2) - 3(-1) - 5 = 6$, so the tangent plane is $4x - 3y - z = 6$.

Part 2: We do exactly the same thing. Set $g_2(x, y, z) = x^2 + 2y^2 + 3z^2 - 6$, then:

$$\nabla g_2 = 2x\partial_x + 4y\partial_y + 6z\partial_z$$

At the point $(1, 1, -1)$, we have $\nabla g_2 = (2, 4, -6)$, therefore the tangent plane is given by $2x+4y-6z = d$. We solve for d by plugging in the point on the plane: $d = 2(1) + 4(1) - 6(-1) = 12$.

Part 3: We are experts at this by now. Set $g_3(x, y, z) = xyz^2 - \log(z-1) - 6$, then

$$\nabla g_3 = yz^2\partial_x + xz^2\partial_y - \frac{1}{z-1}\partial_z$$

The normal to the tangent plane at the point $(-2, -1, 2)$ is given by $n = ((-1)(4), (-2)(1), -1) = (-4, -2, -1)$. The equation of the tangent plane is given by: $-4(-2) - 2(-1) - (2) = 8 = d$ so $4x + 2y + z = -8$.